Zuse Institute Berlin (ZIB)
Freie Universität Berlin
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7. Homework
Numerics for Bioinformaticians
WS 2016/17

Deadline: December 14, 10:00 (**before** the lecture)

The homework should be worked out in groups of two or three students. Each solution sheet must contain the names and student numbers ('Matrikulationnummer') of all group members and the exercise group (Wednesday/Friday). Solutions to homework 1&2 must be handed in in paper form, either hand written or printed out if generated electronically. Please staple all sheets.
Programming tasks must be submitted to max.kleist2@fu-berlin.de or BioInfNumerik@hotmail.com by email. Before sending it, please 'zip' it.

Exercise 1 (Explicit integration, 2.5 points)
Compute the explicit/analytical solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -x_1(t) \cdot k_1 \tag{1}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = x_1(t) \cdot k_1 - x_2(t) \cdot \delta \tag{2}$$

with $x_1(t_0) = C$ and $x_2(t_0) = 0$ and $k_1 > \delta$.
Hint 0: First solve eq. (1), then plug it into eq. (2).

Exercise 2 (Implementation, 1.5 points)
You are given the following ODE to describe the pharmacokinetics (concentration-time profile) of a pharmaceutic drug X in the bloodstream after a single dose D intake:

$$\frac{\mathrm{d}}{\mathrm{d}t}x = k_a\left(\frac{D}{V} \cdot e^{-k_a \cdot t}\right) - k_e \cdot x(t) \tag{3}$$

where k_a and k_e are parameters describing the uptake and elimination of the drug from the body. In our example, let $k_e = 0.3$ and $k_a = 0.5$ and let the dosage of the drug be $D = 200$ [mg] and let the initial concentration in the bloodstream be $x_0 = 0$. For simplicity, let $V = 1$ [L] and thus X be units of [mg/L].
a) Compute the solution of equation (3) numerically using the explicit Euler method with fixed step size $\Delta t = 0.25$ until $t_{final} = 24$. Plot your result at each time instance $t = 0, \Delta t, 2\Delta t, \ldots, 24$. For comparison, put the exact solution using eq. (4) into the same plot (you *should* use a finer time-resolution).
Hint 1: An example of how this should look like is found in Fig. 1 (left).
b) Try $\Delta t = 0.1$, 0.25, 0.5 and compute the average absolute difference between your solution and the exact solution, i.e.

$$\varepsilon = \frac{1}{N}\sum_t |\tilde{x}(t) - x(t)|$$

where $N = t_{final}/\Delta t$ denotes the number of time steps and $\tilde{x}(t)$ is the numeric solution. Plot ε (y-axis) against Δt (x-axis) on a log-log scale (Example: Fig. 1, right). The exact solution is found in eq. (4). Also put a reference line for the order of the error $O(\Delta t)$ in there, i.e. plot Δt vs. Δt.
Hint 2: An example of how this should look like is found in Fig. 1 (right).

Exercise 3 (Implementation, 3 points)
a) Compute the solution of equation (3) numerically using the implicit Euler method with fixed step size $\Delta t = 0.25$ until $t_{final} = 24.$. Plot your result at each time instance $t = 0, \Delta t, 2\Delta t, \dots, 24$. For comparison, put the exact solution for these time instances using eq. (4) into the same plot.
Hint 3: Most solutions and codes in the WWW are incorrect.
b) Try $\Delta t = 0.1$, 0.25, 0.5 and compute the average absolute difference between your solution and the exact solution, i.e.

$$\varepsilon = \frac{1}{N} \sum_t |\tilde{x}(t) - x(t)|$$

where $N = t_{final}/\Delta t$ denotes the number of time steps and $\tilde{x}(t)$ is the numeric solution. Plot ε (y-axis) against Δt (x-axis) on a log-log scale (Example: Fig. 1, right). The exact solution is found in eq. (4). Also put a reference line for the order of the error $O(\Delta t)$ in there, i.e. plot Δt vs. Δt, see Fig. 1 (right).

Exercise 4 (Implementation, 3 points)
a) Compute the solution of equation (3) numerically using the explicit midpoint method with fixed step size $\Delta t = 0.25$ until $t_{final} = 24.$. Plot your result at each time instance $t = 0, \Delta t, 2\Delta t, \dots, 24$. For comparison, put the exact solution for these time instances using eq. (4) into the same plot.
b) Try $\Delta t = 0.1$, 0.25, 0.5 and compute the average absolute difference between your solution and the exact solution, i.e.

$$\varepsilon = \frac{1}{N} \sum_t |\tilde{x}(t) - x(t)|$$

where $N = t_{final}/\Delta t$ denotes the number of time steps and $\tilde{x}(t)$ is the numeric solution. Plot ε (y-axis) against Δt (x-axis) on a log-log scale (Example: Fig. 1, right). The exact solution is found in eq. (4). Also put a reference line for the order of the error in there, i.e. plot Δt vs. $(\Delta t)^2$, see Fig. 1 (right).

Exact solution: The exact solution to equation (3) is given by

$$x(t) = D \left(\frac{k_a}{k_a - k_e} \right) \left(e^{-k_e \cdot t} - e^{-k_a \cdot t} \right). \tag{4}$$

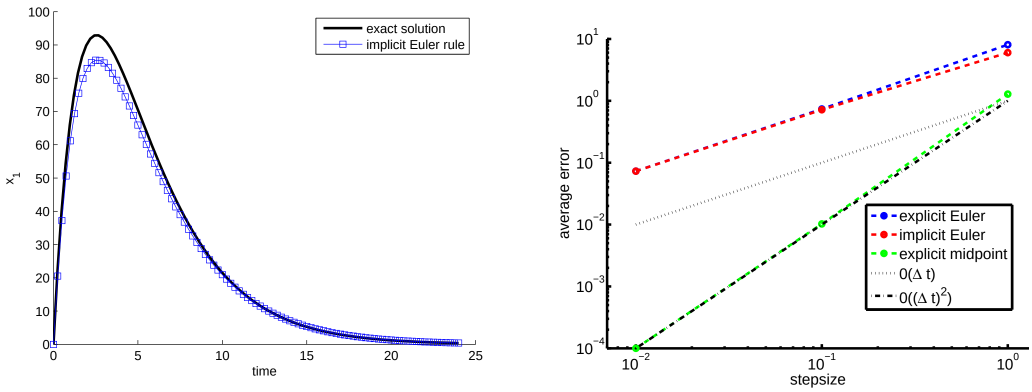

Figure 1: Left:How to plot in Ex.2a, 3a & 4a. Right: How to plot in Ex.2b, 3b & 4b